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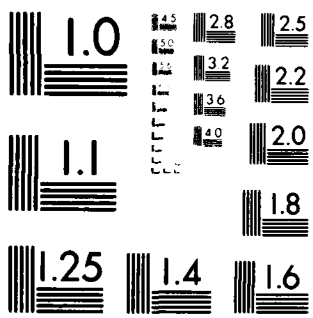

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A RIGHT-INVERSE FOR THE DIVERGENCE OPERATOR IN SPACES OF PIECEWISE POLYNOMIALS. APPLICATION TO THE p-VERSION OF THE FINITE ELEMENT METHOD.

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ABSTRACT

In the first part of this paper we study in detail the properties of the divergence operator acting on continuous piecewise polynomials; more specifically, we characterize the range and prove the existence of a maximal right-inverse whose norm grows at most algebraically with the degree of the piecewise polynomials.

In the last part of this paper we apply these results to the p-version of the Finite Element Method for a nearly incompressible material with homogeneous Dirichlet boundary conditions. We show that the p-version maintains optimal convergence rates in the limit as the Poisson ratio approaches $1/2$. This fact eliminates the need for any "reduced integration" such as customarily used in connection with the more standard h-version of the Finite Element Method.

AMS (MOS) Subject Classifications: 65N30, 73K25

Key Words: discretization of PDE's, finite element method, piecewise polynomial spaces, divergence operators, optimal convergence rates, nearly incompressible materials

Work Unit Number 3 (Numerical Analysis and Computer Science)

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A RIGHT-INVERSE FOR THE DIVERGENCE OPERATOR IN SPACES OF PIECEWISE POLYNOMIALS. APPLICATION TO THE p-VERSION OF THE FINITE ELEMENT METHOD.

Michael Vogelius*

1. Introduction

For many finite element discretizations of P.D.E.'s estimates of the error rely on a detailed study of the divergence operator on corresponding sets of piecewise polynomials. Examples of this are mixed formulations for the Laplace equation or the equations of elasticity (cf. [6],[9]) and discretizations of the Stokes problem (cf. [10]).

Finite element methods in general are based on some partition of the domain and convergent solutions are typically obtained by letting the mesh size h tend to 0 using piecewise polynomials of some fixed degree (the h-version). Recently a different version has received quite a bit of attention (cf. [1,3,4,7]), namely the so-called p-version of the Finite Element Method. The p-version is based on a fixed triangulation and the degree of the piecewise polynomials is increased in order to give convergence.

In section 2 of this paper we analyze the divergence operator with special emphasis on applications for the p-version, to be more precise

(i) we characterize the range of the divergence operator acting on fields that are continuous piecewise polynomials of degree $\leq p+1$ on some triangulation of a domain Ω (for $p \geq 3$).

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(ii) we show that a maximal right-inverse exists, which takes values in continuous piecewise polynomial fields of degree $\leq p+1$, and which has an operator norm that grows at most algebraically with p.

The first of these two results is by far the easiest, it may be obtained from a purely combinatorial argument and use of the Grassmann dimension formula. Since (i) follows very naturally as an intermediate result in our proof of the second (and much harder) assertion we shall only give a sketch of the selfcontained combinatorial argument (cf. Remark 2.1).

The reference [9] contains an explicit construction of a right-inverse for the divergence operator for piecewise polynomials of arbitrarily high degree, but the fields defined that way are not continuous across inter-element boundaries, only the normal components are.

In section 3 we apply the results (i) and (ii) to a particular discretization by the p-version. The continuous model is that of a nearly incompressible linearly elastic material (Poisson ratio $\nu \sim \frac{1}{2}$) with boundary conditions that are homogeneous Dirichlet. We show that the discrete solution converges at an optimal rate even when ν is extremely close to $\frac{1}{2}$; this is in sharp contrast to the h-version where one encounters significantly reduced convergence rates for nearly incompressible materials (we refer to [11] for a more detailed discussion). In [11] we relied on the results (i) and (ii) to analyze the similar problem for the case of natural boundary conditions on a smooth domain.

Let D be a bounded polygonal domain in $\mathbb{R}^n$, n = 1,2. Throughout this paper $H^k(D)$ denotes the standard Sobolev space of functions that have all derivatives of order $\leq k$ in $L^2(D)$.

Whenever no ambiguity is possible the same notation shall also be used for vector valued functions whose components are in $H^k(D)$. Spaces with noninteger exponents are defined by complex interpolation (cf. [5]). The norm on the space $H^s(D)$ is denoted $\|\cdot\|_{s,D}$ and $\mathring{H}^s(D)$ refers to the closure of $C_0^\infty(D)$ in $H^s(D)$.

Ackowledgement

The author would like to thank Professor D. Arnold for many interesting discussions concerning the subject of this paper.

2. A maximal right-inverse for the divergence operator.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and let Σ denote a triangulation of Ω. We assume that Σ has the property that no vertex of a triangle lies in the interior of a side of another.

Before treating piecewise polynomials on the triangulation Σ we establish a few preliminary facts concerning polynomials on a single triangle. In what follows T_0 shall always denote the reference triangle $\{(x,y)|0 < x,\ 0 < y,\ x+y < 1\}$. $\frac{\partial}{\partial n}$ is the derivative along the outward normal to ∂T_0 and $\frac{\partial}{\partial s}$ is the derivative tangential to ∂T_0 and counterclockwise.

Lemma 2.1

Let $Q^{p+1}(x)$ and $R^p(x)$ be two polynomials in x of degree $\leq$ p+1 and $\leq$ p respectively. Assume that 0 and 1 are zeros of multiplicity at least 3 for Q^{p+1} and at least 2 for R^p.

There exists a polynomial $S^{p+1}(x,y)$ in x and y of degree $\leq$ p+1 satisfying

(i) $S^{p+1} = \frac{\partial}{\partial n} S^{p+1} = 0$ on the two sides $x = 0$, $y = 0$,

(ii) $S^{p+1}(x,y) = Q^{p+1}(x)$,

$\frac{\partial}{\partial n} S^{p+1}(x,y) = R^p(x)$ on the side $x+y = 1$.

(iii) $\|S^{p+1}\|_{0,T_0} \leq C(p+1)^K (\|Q^{p+1}\|_{0,[0,1]} + \|R^p\|_{0,[0,1]})$, with constants C and K that are independent of p, Q^{p+1} and R^p.

Proof: Since 0 and 1 are zeros of multiplicity ≥ 2 for both Q^{p+1} and R^p we can write

$$Q^{p+1}(x) = x^2(1-x)^2\, \tilde{Q}^{p-3}(x)$$

$$R^p(x) = x^2(1-x)^2\, \tilde{R}^{p-4}(x) .$$

Define T^{p+1} by

$$T^{p+1}(x,y) = x^2 y^2\, \tilde{Q}^{p-3}(x) .$$

Then

(1) $T^{p+1} = \frac{\partial}{\partial n} T^{p+1} = 0$ on the two sides $x = 0$, $y = 0$.

(2) $T^{p+1}(x,y) = Q^{p+1}(x)$ on the side $x+y = 1$.

By integration we get

$$\iint_{T_0} (T^{p+1}(x,y))^2\, dx\, dy = \int_0^1 x^4 (\tilde{Q}^{p-3}(x))^2 \int_0^{1-x} y^4\, dy\, dx$$

$$= \frac{1}{5}\int_0^1 x^4(1-x)^5(\tilde{Q}^{p-3}(x))^2 dx \leq \frac{1}{5}\int_0^1 (Q^{p+1}(x))^2\, dx ,$$

i.e.,

(3) $$\|T^{p+1}\|_{0,T_0} \leq C\, \|Q^{p+1}\|_{0,[0,1]} .$$

Due to the fact that 0 and 1 are zeros of multiplicity at least 3 for Q^{p+1}, we can also write

$$\frac{\partial}{\partial n} T^{p+1}(x,y)\Big|_{y=1-x} = x^2(1-x)^2\, \tilde{T}^{p-4}(x) \ .$$

Define U^{p+1} by

$$U^{p+1}(x,y) = \sqrt{2}\, x^2y^2\, (x+y-1)(\tilde{R}^{p-4}(x) - \tilde{T}^{p-4}(x)) \ .$$

Then

(4) $\quad U^{p+1} = \frac{\partial}{\partial n} U^{p+1} = 0$ on the two sides $x = 0,\ y = 0$.

(5) $\quad U^{p+1} = 0$ and

$$\frac{\partial}{\partial n} U^{p+1}(x,y) = R^p(x) - \frac{\partial}{\partial n} T^{p+1}(x,y)$$

on the side $x+y = 1$.

By integration as before,

$$\|U^{p+1}\|_{0,\,T_0} \le C(\|R^p\|_{0,[0,1]} + \|\frac{\partial}{\partial n} T^{p+1}\Big|_{y=1-x}\|_{0,[0,1]}) .$$

The standard trace theorem therefore gives

(6) $$\|U^{p+1}\|_{0,T_0} \le C_t(\|R^p\|_{0,[0,1]} + \|T^{p+1}\|_{t,T_0})$$

for any $t > 3/2$.

It is known, and used repeatedly in this paper, that

$$\|T^{p+1}\|_{t,T_0} \le C_t(p+1)^{2t}\, \|T^{p+1}\|_{0,T_0}$$

for any polynomial T^{p+1} of degree $\le$ p+1, and any $t \ge 0$.
(The reference [4] contains a proof of this inequality on a

square. By a localization argument one can extend to other type domains, e.g. triangular). Inserting into (6) and using (3) we thus obtain

(7) $$\|U^{p+1}\|_{0,\,T_0} \le C_t(p+1)^{2t}(\|R^p\|_{0,[0,1]} + \|Q^{p+1}\|_{0,[0,1]})$$

for any $t > 3/2$.

Finally let

$$S^{p+1} = T^{p+1} + U^{p+1} \ .$$

It is then clear from (1),(2),(4) and (5) that S^{p+1} satisfies the requirements (i) and (ii). The estimate (iii) follows from (3) and (7). As K we can use any number > 3. □

Lemma 2.2. Let $Q^p(x)$ and $R^p(x)$ be two polynomials in x of degree $\le p$. Assume that 0 and 1 are zeros of multiplicity at least 2 for Q^p and R^p. Assume furthermore that $\int_0^1 Q^p(x)\,dx = 0$.

There exists a polynomial $S^{p+1}(x,y)$ in x and y of degree $\le p+1$, satisfying

(i) $\frac{\partial}{\partial s} S^{p+1} = \frac{\partial}{\partial n} S^{p+1} = 0$ on the two sides $x = 0$, $y = 0$.

(ii) $\frac{\partial}{\partial s} S^{p+1}(x,y) = Q^p(x)$,

$\frac{\partial}{\partial n} S^{p+1}(x,y) = R^p(x)$ on the side $x+y = 1$.

(iii) $\|S^{p+1}\|_{0,\,T_0} \le C(p+1)^K (\|Q^p\|_{0,[0,1]} + \|R^p\|_{0,[0,1]})$,

with constants C and K that are independent of p, Q^p and R^p .

Proof: Define $\tilde{Q}^{p+1}(x) = -\sqrt{2}\int_0^x Q^p(s)ds$ and construct $S^{p+1}(x,y)$ according to the previous lemma with boundary data $\tilde{Q}^{p+1}$ and R^p. This S^{p+1} has the desired properties. □

Lemma 2.3. Let T be a triangle two sides of which lie on the lines $a_ix + b_iy + c_i = 0$, $i = 1,2$. Let Q^p be a polynomial of the form

$$Q^p(x,y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)^2 R^{p-3}(x,y) \; ,$$

where R^{p-3} is a polynomial in x and y of degree $\leq$ p-3. Assume furthermore that

$$\iint_T Q^p(x,y)\,dx\,dy = 0.$$

There exists a field $\underline{U}^{p+1} = (U_1^{p+1}, U_2^{p+1})$ of polynomials of degree $\leq$ p+1, satisfying

(i) $\underline{U}^{p+1} = 0$ on ∂T

(ii) $\nabla \cdot \underline{U}^{p+1} = Q^p$ in T

(iii) $|\underline{U}^{p+1}|_{1,T} \leq C(p+1)^K |Q^p|_{0,T}$,

with constants C and K that are independent of p and Q^p.

Proof: By an affine transformation T can be mapped onto T_0 in such a way that $a_1x + b_1y + c_1 = 0$ goes to $x = 0$ and $a_2x + b_2y + c_2 = 0$ to $y = 0$. It therefore suffices to prove the result on T_0 with Q^p given by

$$Q^p(x,y) = xy^2 \, R^{p-3}(x,y) \quad .$$

Define $\underline{W}^{p+1}$ as follows

$$W_1^{p+1}(x,y) = \int_0^x Q^p(s,y)\,ds$$

$$W_2^{p+1}(x,y) = 0 \ .$$

This ensures that

$$\underline{W}^{p+1}(x,y) = x^2y^2 \, \tilde{\underline{W}}^{p-3}(x,y) \ ,$$

$$\nabla\cdot\underline{W}^{p+1} = Q^p \text{ and}$$

$$\|\underline{W}^{p+1}\|_{1,\,T_0} \le C(p+1)^2 \, \|Q^p\|_{0,\,T_0} \ ; \tag{8}$$

$\underline{W}^{p+1}$ is therefore 0 on the two sides $x = 0$ and $y = 0$. To make it vanish also along the side $x+y = 1$ we now define an appropriate correction term. Let S^{p+2} denote the polynomial constructed in Lemma 2.2 with boundary data $\underline{n}\cdot\underline{W}^{p+1}(x,1-x)$, $-\underline{s}\cdot\underline{W}^{p+1}(x,1-x)$ for the tangential derivative and the normal derivative respectively (here we use the fact that $\iint_{T_0} Q^p(x,y)\,dx\,dy = 0$). Then

$$\underline{U}^{p+1}(x,y) = \underline{W}^{p+1}(x,y) - \left(\frac{\partial}{\partial y} S^{p+2}, \ -\frac{\partial}{\partial x} S^{p+2}\right)$$

satisfies the requirements (i) and (ii). By integration it also immediately follows that

$$\| \underline{W}^{p+1}|_{y=1-x} \|_{0,[0,1]} \le C \| Q^p \|_{0,T_0} .$$

Combining this with the estimate (iii) of Lemma 2.2 one finds

$$(9) \qquad \| (\frac{\partial}{\partial y} S^{p+2}, -\frac{\partial}{\partial x} S^{p+2}) \|_{1,T_0} \le C(p+1)^4 \| S^{p+2} \|_{0,T_0}$$

$$\le C(p+1)^K \| Q^p \|_{0,T_0}$$

for any $K > 7$. The estimates (8) and (9) easily yield (iii).

□

The following lemma characterizes the range of the divergence operator for the case of a single triangle and polynomial fields with homogeneous boundary conditions. In addition it establishes a bound on the norm of a maximal right-inverse for $\nabla\cdot$. As will be clear from the proof, the latter is the most difficult.

<u>Lemma 2.4</u>. Let T be a triangle the vertices of which have coordinates (x_i, y_i), $1 \le i \le 3$. Let $Q^p(x,y)$ be a polynomial in x and y of degree $\le p$ such that $Q^p(x_i, y_i) = 0$, $1 \le i \le 3$, and $\iint_T Q^p(x,y)\,dx\,dy = 0$.

There exists a field $\underline{U}^{p+1} = (U_1^{p+1}, U_2^{p+1})$ of polynomials of degree $\le p+1$, satisfying

(i) $\underline{U}^{p+1} = 0$ on ∂T

(ii) $\nabla\cdot\underline{U}^{p+1} = Q^p$ in T

(iii) $\| \underline{U}^{p+1} \|_{1,T} \le C(p+1)^K \| Q^p \|_{0,T}$

with constants C and K that are independent of p and Q^p.

Proof: The lemma is trivial for $p \leq 1$, so in the following we shall always assume that $p \geq 2$. Let P^{p+1} denote the polynomials in x and y of degree $\leq p+1$ and let $\mathring{P}^{p+1}$ denote $P^{p+1} \cap \mathring{H}^1(T)$. It is easy to check that the divergence operator maps $\mathring{P}^{p+1} \times \mathring{P}^{p+1}$ into the subspace of P^p characterized by $Q^p(x_i,y_i) = 0$, $1 \leq i \leq 3$, and $\iint_T Q^p(x,y)\,dx\,dy = 0$. The dimension of $\mathring{P}^{p+1} \times \mathring{P}^{p+1}$ is $(p-1)p$. The null space of $\nabla\cdot$ is for $p \geq 4$ given by

$$\{(\frac{\partial}{\partial y} Q^{p+2}, -\frac{\partial}{\partial x} Q^{p+2}) \mid Q^{p+2}(x,y) = \prod_{i=1}^{3} (a_i x+b_i y+c_i)^2 R^{p-4}(x,y),\ R^{p-4} \in P^{p-4}\},$$

where $a_i x + b_i y + c_i = 0$, $1 \leq i \leq 3$, are the equations for the 3 sides of T. For $p = 2,3$ the nullspace consists of 0 only. Hence in general the null space has dimension $(p-3)(p-2)/2$. By using the Grassmann dimension formula we now get that the range of $\nabla\cdot$ must have dimension $(p-1)p - (p-3)(p-2)/2 = (p+1)(p+2)/2 - 4$. Since the subspace of P^p characterized by $Q^p(x_i,y_i) = 0$, $1 \leq i \leq 3$, and $\iint_T Q^p(x,y)\,dx\,dy = 0$ has dimension exactly $(p+1)(p+2)/2 - 4$, we conclude that $\nabla\cdot$ maps $\mathring{P}^{p+1} \times \mathring{P}^{p+1}$ <u>onto</u> this space. In other words, given $Q^p \in P^p$ which satisfies $Q^p(x_i,y_i) = 0$, $1 \leq i \leq 3$, and $\iint_T Q^p(x,y)\,dx\,dy = 0$ one can find $\underline{U}^{p+1}$, a field of polynomials of degree $\leq p+1$, with the properties (i) and (ii).

The estimate (iii) <u>does not</u> follow from the previous argument. The existence of a bounded maximal right-inverse for any fixed p is guaranteed, what we do not know is how the norm depends on p.

In order to prove (iii) it is certainly sufficient to consider $p \geq 5$. Let Q^p be a polynomial of degree $\leq p$ which satisfies $Q^p(x_i,y_i) = 0$, $1 \leq i \leq 3$, and $\iint_T Q^p(x,y)\,dx\,dy = 0$. One can always find a polynomial

R^5 of degree ≤ 5 such that

$$(10)\qquad (\tfrac{\partial}{\partial x})^j \, (\tfrac{\partial}{\partial y})^k \, (Q^p - R^5)(x_i, y_i) = 0$$

for $0 \leq j+k \leq 2$, $1 \leq i \leq 3$,

$$(11)\qquad \iint_T (Q^p - R^5)(x,y)\,dx\,dy = 0 \quad \text{and}$$

$$(12)\qquad |R^5|_{0,T} \leq C|Q^p|_{t,T} \quad \text{for any } t > 3.$$

Due to (10) and (11), $R^5(x_i,y_i) = 0$, $1 \leq i \leq 3$, and $\iint_T R^5(x,y)\,dx\,dy = 0$. The combinatorial argument at the beginning of this proof therefore gives the existence of a field $\underline{v}^6$ with the properties

$$\underline{v}^6 = 0 \quad \text{on } \partial T,$$

$$\nabla\cdot\underline{v}^6 = R^5 \text{ in } T \quad \text{and}$$

$$(13)\qquad |\underline{v}^6|_{1,T} \leq C|R^5|_{0,T}\ .$$

Based on (12) and (13) we get

$$\begin{aligned} |\underline{v}^6|_{1,T} &\leq C|Q^p|_{t,T} \\ &\leq C(p+1)^{2t}\,|Q^p|_{0,T} \end{aligned}$$

for any $t > 3$.

Thus it suffices to prove the assertion (iii) for polynomials Q^p that additionally vanish up to and including second derivatives at the vertices of T. Such a Q^p may, according to Lemma 2.1, be written as

$$Q^p(x,y) = (a_1x + b_1y + c_1)^2 (a_2x + b_2y + c_2)^2 \; Q_1^{p-4}(x,y)$$

$$+ \; (a_2x + b_2y + c_2)^2 (a_3x + b_3y + c_3)^2 \; Q_2^{p-4}(x,y)$$

$$+ \; (a_3x + b_3y + c_3)^2 (a_1x + b_1y + c_1)^2 \; Q_3^{p-4}(x,y) \quad ,$$

where the integral of each term over T vanishes, and the L^2-norm of each term is bounded by

$$C(p+1)^K \, \| Q^p \|_{0,T} \; .$$

An application of Lemma 2.3 term by term now leads to the desired result. □

In order to extend the previous result from a single triangle to the entire triangulation we shall need one more lemma. Let us first introduce some notation. $P^{[p],0}$ denotes the set of continuous functions on Ω, whose restriction to each triangle is given by a polynomial of degree at most p. By an internal vertex of Σ we understand a vertex which lies in Ω (not on $\partial\Omega$) . $P^{[p],-1}$ denotes the set of functions, whose restriction to each triangle is given by a polynomial of degree at most p (no inter-element continuity requirements), with the additional property that:

R.1. "At any internal vertex (x_0,y_0) of the triangulation, where four triangles come together as in Fig. 1 (i.e., the sides meeting at (x_0,y_0) fall on two straight lines) the corresponding values $\{Q_i^p(x_0,y_0)\}_{i=1}^4$ satisfy

$$\sum_{i \text{ even}} Q_i^p(x_0,y_0) = \sum_{i \text{ odd}} Q_i^p(x_0,y_0) \; . \text{"}$$

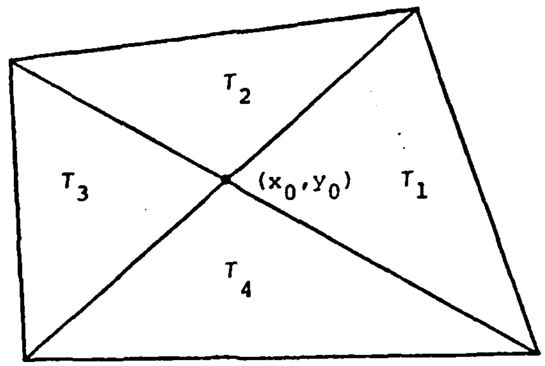

Fig. 1

<u>Lemma 2.5</u>. Let v^p be an element of $P^{[p],-1}$. There exists $\underline{w}^3 \in P^{[3],0} \times P^{[3],0}$ such that

(i) $v^p - \nabla \cdot \underline{w}^3 = 0$ at all vertices of the triangulation Σ .

(ii) $\|\underline{w}^3\|_{1,\Omega} \leq C(p+1)^K \|v^p\|_{0,\Omega}$, with constants C and K that are independent of p, v^p .

Proof: Let us consider two triangles placed adjacent to each other (as in Fig. 2).

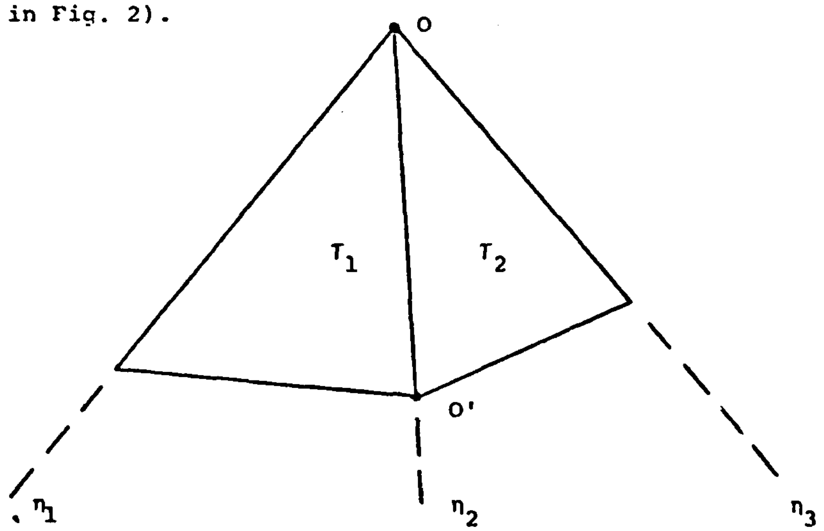

Fig. 2

We easily construct a field $\underline{U}^3$, given by third order polynomials in each triangle, continuous in $\overline{T_1 \cup T_2}$, vanishing along $\partial(\overline{T_1 \cup T_2})$ (the outer boundary) and such that each component has derivative 0 in the direction η_2 at the vertex O' and a prescribed derivative in the direction η_2 at the vertex O. If

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a_{11}\, \eta_1 + a_{12}\, \eta_2$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_{21}\, \eta_1 + a_{22}\, \eta_2 ,$$

then

$$\nabla \cdot \underline{U}^3 = a_{12} \frac{\partial}{\partial \eta_2} U_1^3 + a_{22} \frac{\partial}{\partial \eta_2} U_2^3 \tag{14}$$

in T_1 at the vertex O. Similarly if

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = b_{11}\, \eta_2 + b_{12}\, \eta_3$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = b_{21}\, \eta_2 + b_{22}\, \eta_3 ,$$

then

$$\nabla \cdot \underline{U}^3 = b_{11} \frac{\partial}{\partial \eta_2} U_1^3 + b_{21} \frac{\partial}{\partial \eta_2} U_2^3 \tag{15}$$

in T_2 at the vertex O.

It is easy to see that the vectors $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$ and $\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$ are linearly independent <u>except</u> in case $\eta_3 = -\eta_1$. Since we have the freedom to prescribe $\frac{\partial}{\partial \eta_2} \underline{U}^3$ at the vertex O, (14) and (15) enable us to prescribe any pair of values for $\nabla \cdot \underline{U}^3(O)$, one value in T_1 and another in T_2, provided η_1 and η_3 are linearly independent. If η_1 and η_3 are linearly dependent then the formulas (14) and (15) prescribe the same value for $\nabla \cdot \underline{U}^3(O)$ in T_1 as well as T_2.

Let us now apply this preliminary result to an internal vertex O of Σ.

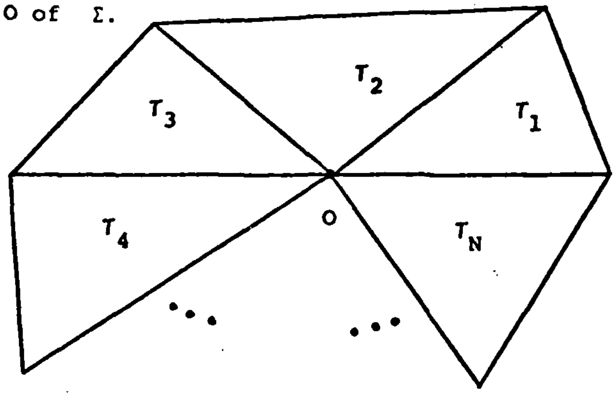

Fig. 3

First suppose that there exists a pair of adjacent triangles such that the combined angle is not π (e.g. T_N and T_1 in Fig. 3). By the procedure just described, we may prescribe the same value for the divergence at O in the two triangles T_1 and T_2. Now we turn counterclockwise and do the same in T_2 and T_3. Continuing that way until we reach the last triangle T_N and performing a summation, we obtain the following set of prescribed values:

$$
\begin{array}{ll}
T_1 & : a_1 \\
T_2 & : a_1 + a_2 \\
T_3 & : a_2 + a_3 \\
 & \vdots \\
T_{N-1} & : a_{N-2} + a_{N-1} \\
T_N & : a_{N-1}
\end{array}
$$

Since the $a_1, \ldots, a_{N-1}$ may be chosen arbitrarily, this clearly gives the possibility of assigning any set of values in the triangles T_1 through T_{N-1}. By assumption T_N and T_1 do not have a combined angle of π so we can also prescribe a value a_N in T_N and 0 in T_1. This permits us to construct any set of values for the divergence at O in the triangles T_1 through T_N.

The only way there can be no two adjacent triangles with combined angle $\neq \pi$ is if the vertex is the intersection of two straight lines (Fig. 1). Taking two triangles at a time as before, we can now prescribe

$$\begin{aligned} T_1 &: a_1 \\ T_2 &: a_1 + a_2 \\ T_3 &: a_2 + a_3 \\ T_4 &: a_3 \end{aligned}$$

i.e., exactly those sets of values that satisfy

$$\sum_{i \text{ even}} \text{value}(T_i) = \sum_{i \text{ odd}} \text{value}(T_i)$$

This is precisely the constraint imposed on $P^{[p],-1}$ at such vertices.

For any $v^p \in P^{[p],-1}$ we are therefore able to find a field $\underline{U}^3$ (continuous in Ω) such that $v^p - \nabla\cdot\underline{U}^3 = 0$ at a particular internal vertex and $\nabla\cdot\underline{U}^3 = 0$ at all other vertices. Our construction has the property that

$$\|\underline{U}^3\|_{1,\Omega} \leq C(p+1)^K \|v^p\|_{0,\Omega}$$

for any $K > 2$.

By a similar construction we can also obtain a $\underline{U}^3$ such that $v^p - \nabla\cdot\underline{U}^3 = 0$ at a particular vertex on $\partial\Omega$ and $\nabla\cdot\underline{U}^3 = 0$ at all other vertices. Adding over all vertices of Σ we get the desired field. $\square$

We are now finally in a position to prove the main result of this section.

Theorem 2.1. Let $\mathcal{D}^p$ denote the space $\nabla\cdot(P^{[p+1],0} \times P^{[p+1],0})$. There exists a linear operator L_p from $\mathcal{D}^p$ into $P^{[p+1],0} \times P^{[p+1],0}$ such that

(i) $\nabla\cdot(L_p v^p) = v^p \qquad \forall\ v^p \in \mathcal{D}^p$

(ii) $\|L_p v^p\|_{1,\Omega} \le C(p+1)^K \|v^p\|_{0,\Omega} \qquad \forall\ v^p \in \mathcal{D}^p$

with constants C and K that are independent of p, v^p.

Furthermore, there exists p_0 such that

$$\mathcal{D}^p = P^{[p],-1}$$

for any $p \ge p_0$.

Proof: We begin by showing that given $v^p \in P^{[p],-1}$ there exists $\underline{u}^{p+1} \in P^{[p+1],0} \times P^{[p+1],0}$ such that

(16) $$\|v^p - \nabla\cdot\underline{u}^{p+1}\|_{0,\Omega} \le C^*(p+1)^{-t^*} \|v^p\|_{0,\Omega} \qquad , \ t^* > 0.$$

It is sufficient to prove (16) for $p \ge 9$. Because of Lemma 2.5 it obviously suffices to consider v^p that satisfy $v^p = 0$ at all vertices of Σ and verify that

(17) $$\|v^p - \nabla\cdot\underline{u}^{p+1}\|_{0,\Omega} \le C(p+1)^{-t} \|v^p\|_{0,\Omega}$$

for some $t > 2$. Construct a piecewise ninth degree polynomial $\tilde{v}^9$ that vanishes up to and including second derivatives on all sides of Σ (and therefore at all vertices) with

$$\iint_T (v^p - \tilde{v}^9)\, dx\, dy = 0$$

on every triangle T. This satisfies $\|\tilde{v}^9\|_{3,\Omega} \le C\|v^p\|_{0,\Omega}$.

Application of Lemma 2.4 separately to each triangle shows that we can find $\underline{w}^{p+1} \in P^{[p+1],0} \times P^{[p+1],0}$ with

(18) $$\nabla\cdot\underline{w}^{p+1} = v^p - \tilde{v}^9$$
$$\|\underline{w}^{p+1}\|_{1,\Omega} \le C(p+1)^K \|v^p\|_{0,\Omega} .$$

It is possible to determine Φ such that

$$\Delta\Phi = \tilde{v}^9 \text{ in } \Omega \ ,$$

and

$$\|\Phi\|_{5,\Omega} \le C\|\tilde{v}^9\|_{3,\Omega} \ .$$

Hence there exist polynomials ϕ^{p+2} of degree $\le p+2$ with

$$\|\Delta(\Phi - \phi^{p+2})\|_{0,\Omega} \le C(p+1)^{-3} \|\tilde{v}^9\|_{3,\Omega}$$
$$\le C(p+1)^{-3} \|v^p\|_{0,\Omega} \ .$$

Therefore $\underline{Y}^{p+1} = \nabla\phi^{p+2}$ satisfies

$$(19) \qquad \|\tilde{v}^9 - \nabla\cdot\underline{Y}^{p+1}\|_{0,\Omega} \le C(p+1)^{-3} \|v^p\|_{0,\Omega} \ .$$

Let $\underline{U}^{p+1}$ be given by $\underline{U}^{p+1} = \underline{W}^{p+1} + \underline{Y}^{p+1}$, a combination of (18) and (19) then yields (17) (with $t = 3$).

It is easily checked that the inclusion $\mathcal{D}^p \subset \mathcal{P}^{[p],-1}$ holds. On the other hand let p_0 be such that $C^*(p_0+1)^{-t^*} < 1$ (C^*, t^* are here the specific constants in (16)) it follows then directly from (16) that the orthogonal complement of $\mathcal{D}^p$ in $\mathcal{P}^{[p],-1}$ contains 0 only, i.e. $\mathcal{D}^p = \mathcal{P}^{[p],-1}$, for $p \ge p_0$.

We now go back to (18); instead of approximating $\tilde{v}^9 \in \mathcal{P}^{[q_0],-1}$, $q_0 = \max\{p_0, 9\}$, we may at this point use the fact that we can find $\underline{Z}^{q_0+1} \in \mathcal{P}^{[q_0+1],0} \times \mathcal{P}^{[q_0+1],0}$ such that

$$\nabla \cdot \underline{Z}^{q_0+1} = \tilde{v}^9$$

and

$$\|\underline{Z}^{q_0+1}\|_{1,\Omega} \le C\|\tilde{v}^9\|_{0,\Omega}$$
$$\le C\|v^p\|_{0,\Omega} \ .$$

Defining $\underline{U}^{p+1} = \underline{W}^{p+1} + \underline{Z}^{q_0+1} \in \mathcal{P}^{[p+1],0} \times \mathcal{P}^{[p+1],0}$ $(p \ge q_0)$ we get

$$\nabla \cdot \underline{U}^{p+1} = v^p , \tag{20}$$

and

$$\| \underline{U}^{p+1} \|_{1,\Omega} \leq C(p+1)^K \| v^p \|_{0,\Omega} . \tag{21}$$

In the construction leading to (20) and (21) we assume that $v^p \in P^{[p],-1}$ satisfies $v^p = 0$ at all vertices of Σ. Applying Lemma 2.5 we obtain that (20) and (21) may be satisfied for any $v^p \in P^{[p],-1}$ $(p \geq q_0)$. Let $L_p v^p$ denote the field in $P^{[p+1],0} \times P^{[p+1],0}$ with minimal H^1-norm for which (20) holds. Then L_p is linear, (i) holds by definition and (ii) follows from (21). This establishes the existence of an apropriate operator L_p provided $p \geq q_0$, and that is obviously sufficient. □

Remark 2.1. If the domain is simply connected then the divergence operator acting on $P^{[p+1],0} \times P^{[p+1],0}$ has as its null space $\{(\frac{\partial}{\partial y} S^{p+2}, -\frac{\partial}{\partial x} S^{p+2}) | S^{p+2}$ is a C^1 piecewise polynomial of degree $\leq p+2\}$. From the formula for the dimension of the space of C^1 piecewise polynomials of degree $\leq q$ $(q \geq 5)$ that is given in [8] one can compute the dimension of this null space in case $p \geq 3$ (the vertices that fall at the intersection of two straight lines play a special role in this formula). By the Grassmann dimension formula one finds that $\mathcal{D}^p = P^{[p],-1}$ for any $p \geq 3$. This argument also extends to cover polygonal domains Ω that are not simply connected, still showing that $\mathcal{D}^p = P^{[p],-1}$ for any $p \geq 3$.

Remark 2.2. Let Ω be an arbitrary bounded Lipschitz domain. We now permit a triangulation of Ω to have triangles with one curved side (in common with $\partial\Omega$).

Assume that any of the triangles with one curved side can be mapped affinely onto a Lipschitz domain of the form

$$\{(x,y) \mid 0 < x < 1,\ 0 < y < g(x)\} \quad ,$$

where g is monotone and satisfies $g(0) = 1$, $g(1) = 0$.

With this assumption one may easily check that Theorem 2.1 remains valid also for triangulations of nonpolygonal domains Ω.

Remark 2.3. The proof of Theorem 2.1 immediately leads to a similar theorem for fields that satisfy homogeneous Dirichlet boundary conditions on a polygonal domain. The space corresponding to $P^{[p],-1}$ will now be defined through the two additional requirements:

R.2. "At any vertex (x_0,y_0) on $\partial\Omega$ where k triangles, $1 \le k \le 4$, come together as in one of the cases shown in Fig. 4 (i.e. the sides meeting at (x_0,y_0) fall on two straight lines),

$$\sum_{\substack{i \text{ even} \\ i \le k}} Q_i^p(x_0,y_0) = \sum_{\substack{i \text{ odd} \\ i \le k}} Q_i^p(x_0,y_0) \ ."$$

R.3. $$\iint_\Omega Q^p(x,y)\,dx\,dy = 0 \ .$$

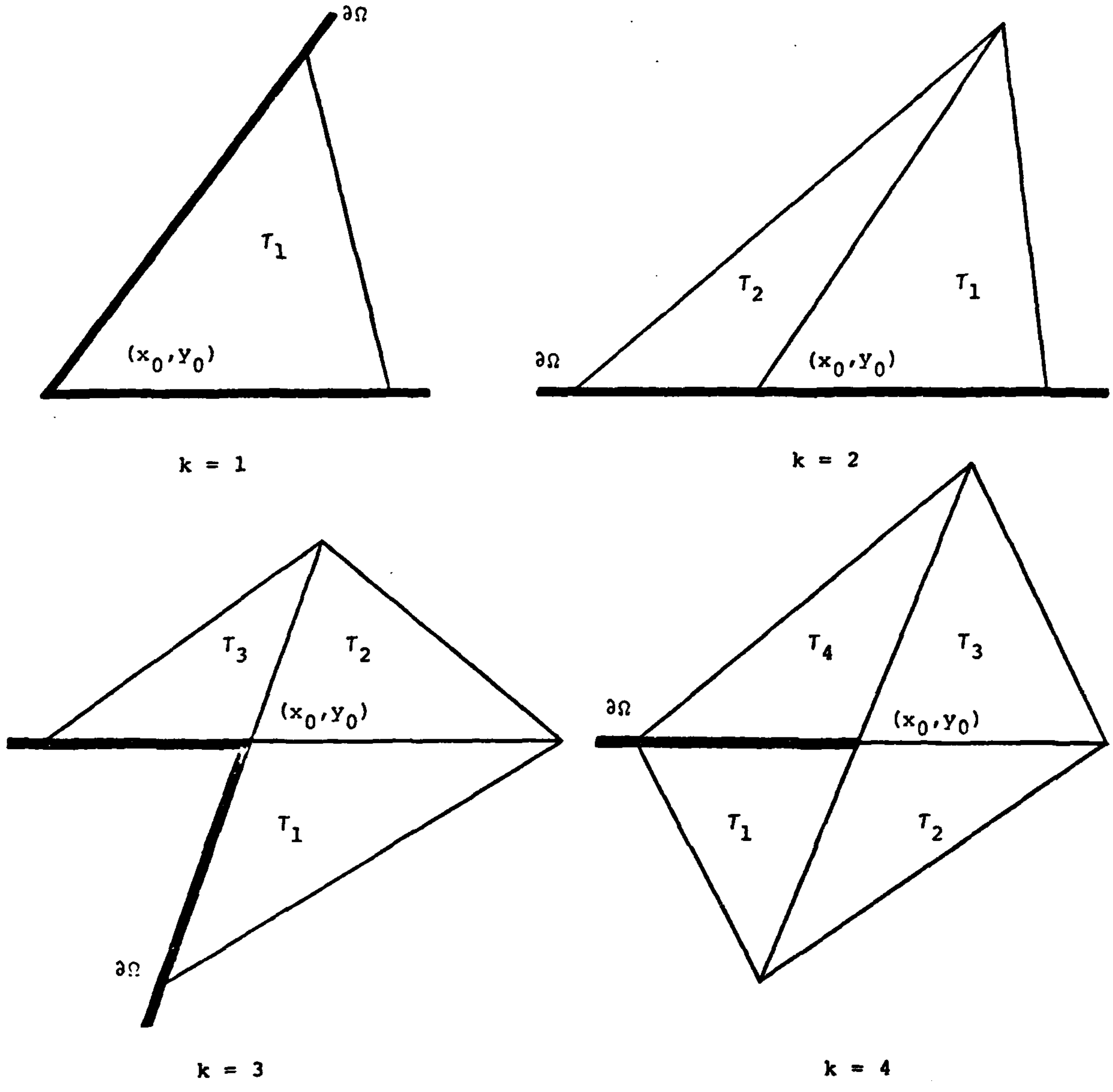

Fig. 4

3. Application to the p-version F.E.M.

In this section we study the behavior of the p-version of the Finite Element Method for the case of a nearly incompressible material. The continuous problem is that of plane strain linear elasticity on a bounded, simply connected polygonal domain Ω. We furthermore assume that all corners of Ω have interior angles $< 2\pi$. Formulated in terms of displacements $\underline{u}_\nu = (u_{\nu,1}, u_{\nu,2})$ the continuous problem is

$$(22) \qquad -\Delta\underline{u}_\nu - \frac{1}{1-2\nu}\nabla(\nabla\cdot\underline{u}_\nu) = \underline{F} \text{ in } \Omega,$$

where $\underline{F}$ represents an external force (to be specific $\underline{F}$ is the external force scaled by $2(1+\nu)/E$). $E > 0$ denotes the Young modulus and $0 < \nu < \frac{1}{2}$ the Poisson ratio; the case that ν is very close to $\frac{1}{2}$ corresponds to a nearly incompressible material. The boundary conditions are

$$(23) \qquad \underline{u}_\nu = 0 \quad \text{on } \partial\Omega .$$

The analysis presented here could easily be extended to the case that homogeneous Dirichlet data is given on part of the boundary and natural (stress) boundary conditions on the rest.

Solving the boundary value problem (22) and (23) is equivalent to minimizing the energy expression

$$(24) \qquad \iint_\Omega \left[\left(\frac{\partial}{\partial x} v_1\right)^2 + \left(\frac{\partial}{\partial y} v_2\right)^2 + \frac{1}{2}\left(\frac{\partial}{\partial x} v_2 + \frac{\partial}{\partial y} v_1\right)^2 + \frac{\nu}{1-2\nu}(\nabla\cdot\underline{v})^2 \right] dx\, dy$$

$$- \iint_\Omega \underline{F}\ \underline{v}\ dx\, dy$$

in $\overset{\circ}{H}{}^1(\Omega)$. (We minimally assume that $\underline{F} \in H^{-1}(\Omega)$, the dual of $\overset{\circ}{H}{}^1(\Omega)$.)

For values of ν very close to ½ (22) and (23) may be viewed as a penalized version of the Stokes problem

$$(25) \qquad \begin{aligned} -\Delta\underline{U} + \nabla P &= \underline{F} \quad \text{and} \\ \nabla\cdot\underline{U} &= 0 \quad \text{in } \Omega \text{, with} \\ \underline{U} &= 0 \quad \text{on } \partial\Omega . \end{aligned}$$

(It is well known that (25) has a unique solution $\underline{U} \in \overset{\circ}{H}{}^1(\Omega)$, $P \in L^2(\Omega)/\{\text{constants}\}$ provided $F \in H^{-1}(\Omega)$, cf. [10].)

We shall make use of the following simple estimate

$$(26) \qquad \|\underline{u}_\nu - \underline{U}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{u}_\nu - \underline{U})\|_{0,\Omega} \le C(1-2\nu)^{1/2} \|P\|_{0,\Omega} ,$$

where $\|P\|_{0,\Omega}$ refers to the norm in the quotient space.

Let $\Sigma = \{T_i\}_{i=1}^M$ be a triangulation of Ω with the property that no vertex of a triangle lies in the interior of a side of another. $\overset{\circ}{P}{}^{[p],0}$ denotes the set of continuous functions, whose restriction to each triangle is a polynomial of degree at most p and whose trace on $\partial\Omega$ vanishes. The finite dimensional problems we consider are those that arise when the energy expression (24) is minimized in $\overset{\circ}{P}{}^{[p],0} \times \overset{\circ}{P}{}^{[p],0}$, $p \ge 0$. For values of ν away from ½ the finite dimensional solutions are easily seen to be as good as the best H^1-approximations to $\underline{u}_\nu$ from within the spaces $\overset{\circ}{P}{}^{[p],0} \times \overset{\circ}{P}{}^{[p],0}$.

The aim of this section is to prove an optimal asymptotic error estimate in terms of negative powers of p, which is valid in the limit as ν approaches ½. First a simple approximation result.

Lemma 3.1. For any $k \geq 1$ and any $\varepsilon > 0$ there exists a constant $C_{k,\varepsilon}$ such that

$$\inf_{\underline{Q}^p \in \overset{\circ}{P}^{[p],0} \times \overset{\circ}{P}^{[p],0}} \|\underline{v} - \underline{Q}^p\|_{1,\Omega} \leq C_{k,\varepsilon}(p+1)^{-k+1+\varepsilon}\|\underline{v}\|_{k,\Omega}$$

for any $p \geq 0$ and any $\underline{v} \in (H^k(\Omega) \cap \overset{\circ}{H}{}^1(\Omega))^2$.

Proof: see [4] .

Let $\underline{u}_{\nu,p}$ denote the element of $\overset{\circ}{P}^{[p],0} \times \overset{\circ}{P}^{[p],0}$ that minimizes (24), we shall now prove that $\underline{u}_{\nu,p}$ converges to $\underline{u}_\nu$ at the optimal rate even in the limit as ν approaches ½. Unlike the result proven in [11], for the case of a smooth boundary $\partial\Omega$ and natural boundary conditions on all of $\partial\Omega$, the following theorem does not establish uniformly valid optimal convergence rates. The weaker formulation is due to the nonsmoothness of the boundary $\partial\Omega$. Even for a fixed load $\underline{F}$ the corner singularities and therefore the smoothness of the solution $\underline{u}_\nu$ is likely to depend on ν . This means that there are no uniformly valid optimal rates, rather the optimal rates depend on ν. It is undoubtedly possible to establish error estimates that exhibit these optimal rates (depending on ν) but with constants that are independent of ν .Technically however this introduces a lot of difficulties that are irrelevant to the main scope of this section, namely to understand exactly what makes the p-version superior for nearly incompressible materials.

<u>Theorem</u> 3.1. Assume that $s \geq 1$ and $\nu \in]0,1/2[$. Let $\underline{u}_\nu$ be the solution to (22) and (23), and let $\underline{U}$, P be the solution to (25). For any $p \geq 0$, $\underline{u}_{\nu,p}$ denotes the element in $\overset{\circ}{P}^{[p],0} \times \overset{\circ}{P}^{[p],0}$ that minimizes (24).

Given $\varepsilon > 0$ there exists $C_{s,\varepsilon}$ (independent of ν and p) such that whenever $\underline{U} \in H^s(\Omega)$,

$$\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega}$$

$$\leq C_{s,\varepsilon}(p+1)^{-s+1+\varepsilon} \|\underline{U}\|_{s,\Omega} + C(1-2\nu)^{1/2} \|P\|_{0,\Omega}$$

(the constant C is independent of all parameters).

<u>Proof</u>: Let $T_{\nu,p}(\underline{v})$ denote the projection of $\underline{v}$ onto $\overset{\circ}{P}^{[p],0} \times \overset{\circ}{P}^{[p],0}$ in the energy inner product associated with (24). Then $\underline{u}_{\nu,p} = T_{\nu,p}(\underline{u}_\nu)$ and therefore

$$\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{\frac{1}{2}} \|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega}$$

$$\leq \|\underline{u}_\nu - \underline{U}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{\frac{1}{2}} \|\nabla\cdot(\underline{u}_\nu - \underline{U})\|_{0,\Omega}$$

$$+ \|\underline{U} - T_{\nu,p}(\underline{U})\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{\frac{1}{2}} \|\nabla\cdot(\underline{U} - T_{\nu,p}(\underline{U}))\|_{0,\Omega}$$

$$+ \|T_{\nu,p}(\underline{U} - \underline{u}_\nu)\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{\frac{1}{2}} \|\nabla\cdot(T_{\nu,p}(\underline{U} - \underline{u}_\nu))\|_{0,\Omega} .$$

The operators $T_{\nu,p}$ are uniformly bounded in the norms $\|\ \|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{\frac{1}{2}} \|\nabla\cdot(\ )\|_{0,\Omega}$ and by using the estimate (26) we thus conclude

$$\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{u}_\nu-\underline{u}_{\nu,p})\|_{0,\Omega}$$

$$\text{(27)} \qquad \leq \|\underline{U} - T_{\nu,p}(\underline{U})\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{U} - T_{\nu,p}(\underline{U}))\|_{0,\Omega}$$

$$+ C(1-2\nu)^{1/2} \|P\|_{0,\Omega}.$$

Similar to Lemma 2.2 in [11] (for which we did not use the smoothness of the domain Ω) we get for any $\underline{v}$ with $\nabla\cdot\underline{v} = 0$ that

$$\|\underline{v} - T_{\nu,p}(\underline{v})\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{v} - T_{\nu,p}(\underline{v}))\|_{0,\Omega}$$

$$\leq D(C_p)^2 \left(\inf_{\underline{Q}^p \in \mathring{P}^{[p],0} \times \mathring{P}^{[p],0}} \|\underline{v} - \underline{Q}^p\|_{1,\Omega} \right),$$

where C_p denotes the $L(L^2,H^1)$-norm of any maximal right-inverse for the divergence operator acting on $\mathring{P}^{[p],0} \times \mathring{P}^{[p],0}$ (here we assume that $p \geq 3$ so that $\nabla\cdot(\mathring{P}^{[p],0} \times \mathring{P}^{[p],0}) \neq 0$). Theorem 2.1 together with Remark 2.3 shows that we can obtain $C_p \leq C(p+1)^K$, and as a consequence of this and Lemma 3.1,

$$\|\underline{v} - T_{\nu,p}(\underline{v})\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{\frac{1}{2}} \|\nabla\cdot(\underline{v} - T_{\nu,p}(\underline{v}))\|_{0,\Omega}$$

(28)

$$\leq C_{k,\varepsilon}(p+1)^{2K-k+1+\varepsilon/2} \|\underline{v}\|_{k,\Omega},$$

provided $\underline{v} \in H^k(\Omega) \cap \mathring{H}^1(\Omega)$ and $\nabla\cdot\underline{v} = 0$.

The operators $T_{\nu,p}$ are uniformly bounded, i.e.,

$$\text{(29)} \qquad \|\underline{v} - T_{\nu,p}(\underline{v})\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{\frac{1}{2}} \|\nabla\cdot(\underline{v} - T_{\nu,p}(\underline{v}))\|_{0,\Omega}$$

$$\leq C \|\underline{v}\|_{1,\Omega}$$

provided $\nabla\cdot\underline{v} = 0$.

Let $\mathcal{H}^r(\Omega)$ denote the space $H^r(\Omega) \cap \mathring{H}^1(\Omega) \cap \{\underline{v} \mid \nabla\cdot\underline{v} = 0\}$, $r \geq 1$. Since Ω is simply connected $\mathcal{H}^r(\Omega)$ may also be characterized as $\{(\frac{\partial}{\partial y}\phi, -\frac{\partial}{\partial x}\phi) \mid \phi \in H^{r+1}(\Omega) \cap \mathring{H}^2(\Omega)\}$, i.e., $\mathcal{H}^r(\Omega)$ is isomorphic to $H^{r+1}(\Omega) \cap \mathring{H}^2(\Omega)$. Using the same ideas as in [2] (see also [12]) only with the biharmonic operator instead of the Laplace operator one gets the following result concerning complex interpolation and the spaces $H^{r+1}(\Omega) \cap \mathring{H}^2(\Omega)$:

$$(H^{r+1}(\Omega) \cap \mathring{H}^2(\Omega),\ H^{t+1}(\Omega) \cap \mathring{H}^2(\Omega))_\theta = H^{r+1+(t-r)\theta}(\Omega) \cap \mathring{H}^2(\Omega)$$

for any $1 \leq r \leq t$ and $0 < \theta < 1$ (even though Ω is not smooth). Due to this result and the aforementioned isomorphism we can now interpolate between (28) and (29) to obtain

$$\begin{aligned} &\|\underline{v} - T_{\nu,p}(\underline{v})\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{v} - T_{\nu,p}(\underline{v}))\|_{0,\Omega} \\ (30)\qquad &\leq C_{k,\varepsilon,\theta}(p+1)^{2K\theta-(k-1)\theta+\varepsilon/2} \|\underline{v}\|_{1+(k-1)\theta,\Omega} \end{aligned}$$

for any $0 < \theta < 1$ and any $\underline{v} \in \mathcal{H}^{1+(k-1)\theta}(\Omega)$. Choose $k > 4K\cdot\frac{s-1}{\varepsilon} + 1$ and define $\theta = \frac{s-1}{k-1}$. The estimate (30) with $\underline{U}$, the solution to (25), inserted for $\underline{v}$ then reads

$$\begin{aligned} &\|\underline{U} - T_{\nu,p}(\underline{U})\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{U} - T_{\nu,p}(\underline{U}))\|_{0,\Omega} \\ (31)\qquad &\leq C_{s,\varepsilon}(p+1)^{-s+1+\varepsilon} \|\underline{U}\|_{s,\Omega} . \end{aligned}$$

Combining (27) and (31) we get the desired estimate. □

Remark 3.1. An alternate way to formulate the result of Theorem 3.1 is, in light of the estimate (26),

$$\|\underline{U} - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{U} - \underline{u}_{\nu,p})\|_{0,\Omega}$$
$$\leq C_{s,\varepsilon}(p+1)^{-s+1+\varepsilon} \|\underline{U}\|_{s,\Omega} + C(1-2\nu)^{1/2} \|P\|_{0,\Omega} .$$

By choosing the penalty parameter $\frac{\nu}{1-2\nu}$ sufficiently large and minimizing (24) in $\mathring{P}^{[p],0} \times \mathring{P}^{[p],0}$ we thus obtain a solution $\underline{u}_{\nu,p}$ that converges to $\underline{U}$, the solution of the Stokes problem, at the rate $(p+1)^{-s+1}$ (modulo an ε) provided $\underline{U} \in H^s(\Omega)$. This rate is optimal for a general $\underline{U} \in H^s(\Omega)$; if namely

$$\|\underline{U} - \underline{v}_j\|_{1,\Omega} \leq C(p_j+1)^{-s+1-\delta}$$

for some $\delta > 0$ and some sequences $\underline{v}_j \in P^{[p_j],0} \times P^{[p_j],0}$, $p_{j+1}/p_j \leq \Lambda < \infty$, $p_j \uparrow \infty$, then one could conclude (cf. [4]) that

$$\underline{U} \in H^{s+t}_{loc}(T_i)$$

for any $t < \delta$ and any triangle $T_i \in \Sigma$. ($\underline{U} \in H^{s+t}_{loc}(T_i)$ means that $\underline{U} \in H^{s+t}(K)$ on all compact subsets K of T_i.)

Remark 3.2. Whether one thinks of $\underline{u}_{\nu,p}$ as an approximation to the displacement for a nearly incompressible material or as an approximation to the solution of the Stokes problem, the behavior of the error is remarkably different from that of more standard finite element discretizations. If Σ_h denotes a triangulation of mesh size h and one minimizes (24) in the set of continuous piecewise polynomials of some fixed degree for decreasing values of h, then it is well known that the convergence rates obtained this way are not optimal as ν approaches $\frac{1}{2}$, for piecewise linear functions there is no convergence at all. We refer to the introduction of [11] for more details.

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discretization of PDE's, finite element method, piecewise polynomial spaces, divergence operators, optimal convergence rates, nearly incompressible materials

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In the first part of this paper we study in detail the properties of the divergence operator acting on continuous piecewise polynomials; more specifically, we characterize the range and prove the existence of a maximal right-inverse whose norm grows at most algebraically with the degree of the piecewise polynomials.

(cont.)

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ABSTRACT (cont.)

In the last part of this paper we apply these results to the p-version of the Finite Element Method for a nearly incompressible material with homogeneous Dirichlet boundary conditions. We show that the p-version maintains optimal convergence rates in the limit as the Poisson ratio approaches 1/2. This fact eliminates the need for any "reduced integration" such as customarily used in connection with the more standard h-version of the Finite Element Method.

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